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DYNAMICS OF THE FRACTURE OF UNIDIRECTIONAL GLASS – PLASTIC

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UDC 539.3+539.4

The dynamic problem of the concentration of stresses and the subsequent propagation of a flaking crack in unidirectional glass–plastic is considered. A plane deformation is investigated and the glass–plastic material corresponds to the model considered in [1], i.e., it is assumed that the armoring of the fiber is in the uniaxial stressed state (extension–compression), and the filler (binding) is subjected only to a shear stress.

For practical purposes it is important to explain the features of the kinetics of cracks and the possibility of localizing them. In this paper we solve these problems by a numerical method, which enables us, with acceptable accuracy, to describe the nonstationary wave process of stress concentration and subsequent fracture.

The problem of the dynamic concentration of stresses in the region of a defect in a glass–plastic is considered in a limited number of papers (see, e.g., [2, 3]). Here we use the formulation of the problem given in [2], where the solution of the dynamic problem is obtained in the form of the sum of a series with a finite number of terms, each of which corresponds to the contribution of a wave reflected from a certain fiber. In [3] the problem of approximating the dynamic solution to the static solution with time is discussed. It is not possible to analyze the kinetics of fracture using analytical methods.

The formulation of the problem is as follows: A fiber is stretched to infinity with a constant force; at zero instant of time, due to a certain defect, one of the fibers instantaneously fractures; then the broken fiber begins to be unloaded, while the load on all the others is increased, the perturbations from one rod to another being transferred by shear waves into the binder; if we assume that the increase in the load on all the fibers does not lead to their fracture, fracture can only occur in the form of longitudinal flaking cracks.

We will direct the y coordinate along the fiber, and the x coordinate perpendicular to it, and we will take the origin of coordinates at the defect. We will take as the unit of measurement the quantities which relate to the filler: the density ρ, the shear modulus G, the velocity of shear waves $c_2 = \sqrt{G/\rho}$, and the distance between fibers H (H/c_2 is the unit of time). We will introduce the following notation: ρ_1, E, and h are the density, Young's modulus, the thickness of the fiber ($c_1 = \sqrt{E/\rho_1}$ is the velocity of sound in the fiber), $u_j(y, t)$ is the displacement of the j-th fiber ($j = 0, \pm 1, \pm 2, \ldots$), $v(x, y, t)$ is the displacement of a point of the filler, $\sigma_j = E\partial u_j/\partial y$, $\tau = G\partial v/\partial x$ are the stresses in the fibers and the binder.

The glass–plastic is stretched to infinity with a stress P. We will solve the problem with respect to additional perturbations due to breaking of the fiber (suppose this is the fiber $j = 0$). The equations in the displacements and the boundary conditions have the form (the initial conditions are the zero conditions)

$$\partial^2 v/\partial t^2 = \partial^2 v/\partial x^2; \tag{1}$$

$$\frac{1}{c_1^2}\frac{\partial^2 u_j}{\partial t^2} = \frac{\partial^2 u_j}{\partial y^2} + \frac{1}{Eh}Q_j, \quad Q_j = \tau|_{x=j+0} - \tau|_{x=j-0}; \tag{2}$$

$$v(j, y, t) = u_j(y, t); \tag{3}$$

$$\partial u_j/\partial t = -P/E\ (j = 0),\ u_j = 0\ (j \neq 0) \text{ for } y = 0, \tag{4}$$

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where Q_j is the force acting on the j-th fiber from the binder side.

We will assume that the binder begins to break after the shear stresses exceed a certain value τ_p. The criterion for fracture to occur at a certain point of the binder is the inequality

$$\tau(x, y, t) > \tau_p. \tag{5}$$

Since we have assumed that fibers with numbers $j \neq 0$ do not break, the intensity of the shear stresses in the region $|x| > 1$ is small compared with the zone $|x| \leq 1$. Hence, we will confine ourselves to investigating the fracture by studying the flaking process in the zone $|x| \leq 1$.

In view of the symmetry we will consider a quarter of the plane. We will divide the possible regions of fracture into three (this subdivision is purely conventional and is introduced for the sake of convenience): The boundaries of the zone are $x = 0$ and $x = 1$, and the region $0 < x < 1$. We will consider how the equations of the problem change if fracture occurs in any of these three regions.

1) The boundary $x = 0$. If at a certain point of the binder $x = +0$, $y = y_{0*}$ at the instant of time t_{0*} condition (5) is satisfied, in Eq. (2) we must put

$$Q_0 = 0 \quad (y = y_{0*},\ t = t_{0*}).$$

At subsequent instants of time $(t > t_{0*})$ the following versions of the development of a fracture are possible:

a) The front of the fracture moves along the fiber: Flaking occurs between the binder and the fiber and cracks are formed $l_0^-(t) \leq y \leq l_0^+(t)$, where l_0^- and l_0^+ are the coordinates of the ends of the cracks.

In this case we must put in Eq. (2)

$$Q_0 = 0 \quad (y \in [l_0^-,\ l_0^+],\ t > t_{0*}), \tag{6}$$

while the boundary condition for the binder (3) must be replaced by the condition that there are no stresses on the free surface formed

$$\tau = 0 \quad (x = +0,\ y \in [l_0^-,\ l_0^+],\ t > t_{0*}). \tag{7}$$

b) When flaking occurs (or without it) the fracture front also moves into the depth of the binder in the direction $x > 0$. In this case, condition (6) for the fiber is not changed, but it is necessary to impose condition (7) on the boundary between the fractured and unfractured regions of the binder, replacing the boundary condition (3) by the condition that there should be no stresses on the free surface in front of the fracture front $x = x^*(t, y)$

$$\tau = 0 \quad (x = x^*,\ y \in [L_0^-(x),\ L_0^+(x)],\ t > t_{0*}). \tag{7'}$$

It should be noted that the coordinates of the ends $L_0^-(x)$, $L_0^+(x)$ in (7') may not be the same as l_0^-, l_0^+ in (7).

2) The boundary $x = 1$. As in case 1) we have [below we will not draw any particular distinction between versions a) and b), corresponding to case 1)]

$$Q_1 = \tau|_{x=1+0} \quad (y \in [l_1^-,\ l_1^+],\ t \geqslant t_{1*}); \tag{8}$$

$$\tau = 0 \quad (x = 1-0,\ y \in [l_1^-,\ l_1^+],\ t \geqslant t_{1*}), \tag{9}$$

where t_{1*} is the beginning of fracture, and $l_1^-(t)$, $l_1^+(t)$ are the coordinates of the ends of the crack where the binder flakes away from the first whole fiber.

3) Suppose $A_k(x, y)$ are points inside the zone $0 < x < 1$, in which at the instants of time t_{*k} condition (5) is achieved. At these points when $t \geq t_{*k}$, Eq. (1) no longer makes sense, and instead we must introduce the condition

$$\tau = 0 \ (\overline{x, y} = A_k(x, y),\ t \geqslant t_{*k}), \tag{10}$$

which will serve as the boundary conditions for Eq. (1), determined for the whole material $(\overline{x, y} \neq A_k(x, y))$.

Expressions (7), (9), and (10) on the free boundary formed in the binder by the flaking cracks or by the fracture front were written on the assumption that the conditions under which the model used is applicable are satisfied (the deformations are shear deformations and they are small) and, in addition, during fracture no reversible phenomena occur on the surfaces formed, in other words, the law of state for the binder is assumed to hold at all points of the latter including the boundary.

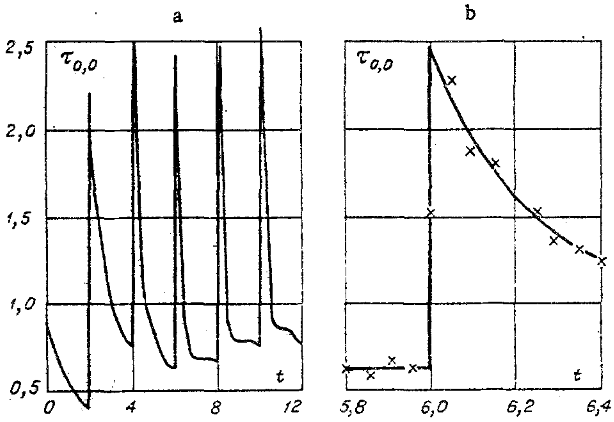

Fig. 1

An investigation of the processes that occur in the region occupied by the fractured material is outside the framework of this paper and will not be considered here.

We will note some characteristic features of the behavior of the desired solution. When the wave process propagates from the neighborhood of the defect into the depth of the glass–plastic material a complex interference pattern is produced. The main portion of the energy is transferred by the shear waves, which, being partially reflected from the fiber as from a movable boundary, transfer part of the energy into longitudinal motions of the fibers. The latter in turn radiate shear waves into the binder, and the region occupied by the perturbations increases, until at a certain (large) distance from the region of the defect (formerly at infinity) the wave process is practically attenuated.

The stress waves in the fibers and the binder have a multistep form, sections of smooth change of the solution alternate with discontinuities, conveyed by the reflected-wave fronts, and the amplitudes of the jumps on the fronts, due to multiple interaction with one another, do not have an ordered structure.

Equations (1) and (2) with boundary conditions (3) and (4) and the additional conditions as regards the flaking (5)-(10), which describe this picture, were solved numerically.

In order to describe the wavefronts with acceptable accuracy the parameters of the difference grid (Δt, Δx, Δy) were chosen in order to minimize the numerical dispersion [5]. In the plane problem of the dynamic theory of elasticity it is quite difficult to minimize the numerical dispersion. The model of the medium investigated is a certain degenerate case of the plane problem. It differs from the latter in that although the fronts of the longitudinal and shear waves propagate with the same velocities c_1 and c_2, these two types of waves interact only along the straight lines determined by the fibers, and not over the whole region occupied by the perturbations. Due to the properties of the model used it is possible to minimize the numerical dispersion quite effectively.

On the basis of a harmonic analysis it can be shown that the group velocities of the high-frequency components (in the limit $\omega \to \infty$), forming the fracture, will be identical in the differential and difference models, if the parameters of the grid satisfy the following condition:

$$\Delta t = \Delta x = c_1^{-1} \Delta y. \tag{11}$$

In this case, the region of dependence of the finite-difference equations coincides with the region of dependence of the initial equations (1) and (2). Equation (11), however, contradicts the stability conditions of the explicit scheme obtained in the form of the following inequalities:

$$\Delta t \leqslant \Delta x,\ c_1 \Delta t \leqslant \Delta y (1 + \Delta x/\alpha)^{-1/2},\ \alpha = \rho_1 h, \tag{12}$$

where α is the mass of the fiber referred to the mass of the binder. In order to minimize the numerical dispersion and to satisfy the stability conditions, we must designate the limiting values of the inequality (12) for fixed Δt with spatial steps Δx and Δy

$$\Delta x = \Delta t,\ \Delta y = c_1 \Delta t (1 + \Delta t/\alpha)^{1/2}. \tag{13}$$

It should be noted that the numerical dispersion which remains in this case in the difference algorithm is introduced solely by the wave processes which propagate in the fibers.

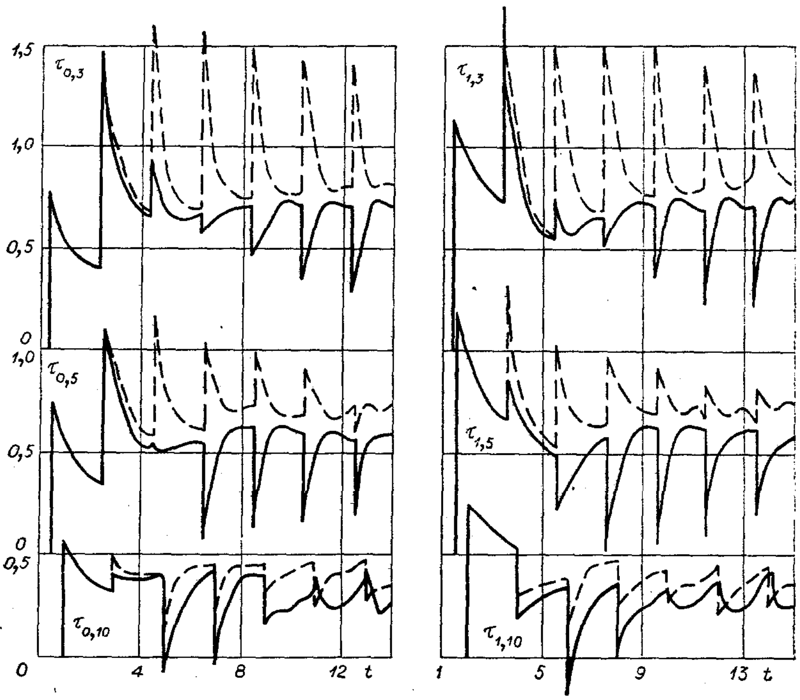

Fig. 2

When choosing the grid in the form (13) two independent parameters remain in the numerical algorithm, namely, the parameter of the grid Δt (or Δx) and the parameter α, characterizing the inertial properties of the glass–plastic. By reducing the value of Δt for fixed α, we can ensure that the effects of numerical dispersion do not, in practice, appear. At the same time, however, when Δt is reduced the volume of the computer memory required to investigate the wave process in times sufficient for fractures to develop and become localized increases. Hence, it is necessary to choose a certain optimum value for Δt. Calculations showed that when $\Delta x/\alpha = \Delta t/\alpha \leq 0.025$ the numerical and analytical results are in fairly good agreement: The intensities of the fronts can be calculated with an error of not more than 5% (calculations with $\Delta t/\alpha = 0.005$ give a disagreement only in the third–fourth places). The above related to calculations of the discontinuities, the smooth components of the solution can be accurately calculated in practice.

In Fig. 1a for $\alpha = 2$ and $\Delta t = 0.05$ we show the shear stresses at a point of the binder closest to the point where the zeroth fiber fractures ($\tau_{0,0} = \tau|_{x=0,\ y=0}$). The numerical and analytical results are practically identical (they are not distinguishable on the scale shown), but some disagreement is observed in the small neighborhood of even values of t – in the region of the fractures. The nature of the disagreement is shown in Fig. 1b (the continuous line is the analytical solution [3], and the crosses are the numerical solution).

To economize on the computer memory the rectangular region, occupied by the glass–plastic, was replaced by a triangular region with a fictitious boundary on the hypotenuse. The position of this boundary depended on the value of T ($T = n\Delta t$, where n is the number of time steps) and was chosen so that during the time T the waves reflected from the fictitious boundary do not reach the region in which the process of interest is calculated.

We will present some of the calculation data and analyze them. In Fig. 2, for $\tau_p = 1.8$, and $\alpha = 2$ ($\Delta t = 0.05$) the dashed lines show the stresses $\tau_{x,y}$ ($x = 0.1$; $y = 3\Delta y, 5\Delta y, 10\Delta y$) after flaking has occurred. The flaking cracks cease to grow when $t = 4$. Their parameters are $t_{0*} = 0.05$, $t_{1*} = 1.05$, $y_{0*} = y_{1*} = l_0^- = l_1^- = 0$, $l_0^+ = l_1^+ = 3\Delta y$. The continuous lines show the stresses when there is no fracture ($\tau_p > \tau_{max}$, where τ_{max} are the maximum stresses in the binder).

It should be noted that the sign of the fracture at $\tau_{x,y}$ for $y > 0$ changes with time, and the envelope of the sharp peaks is a smooth alternating sign function which decreases as t increases. In turn, as y increases the maximum value of the fracture also decreases and the stresses approach the static value fairly rapidly (for their y values). These static values are the same for $\tau_{0,y}$ and $\tau_{1,y}$ [4]. It can be seen from Fig. 2 that an essential difference in the operation of the material on flaking and without this is observed at a comparatively small distance from the fracture zone. This difference is characterized by the fact that when flaking

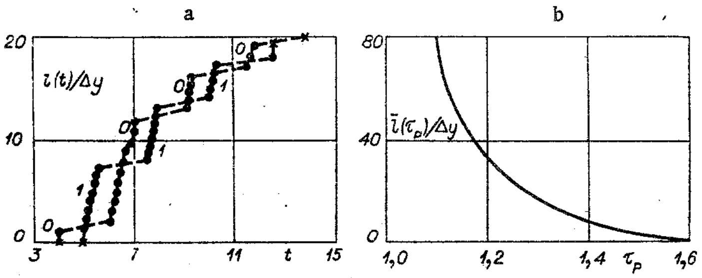

Fig. 3

occurs the peaks of the stresses are much higher, and the discontinuities change sign for large values of time. The qualitative picture of the stress distribution remains the same. It can be seen that for $y = 10\Delta y$ the difference between these two cases is comparatively small.

In Fig. 3a, for $\alpha = 20$, $\Delta t = 0.1$, and $\tau_p = 1.25$ we show the development with time of the flaking cracks of the binder from the torn fiber 0 and the first whole fiber 1, the crosses showing the ends of the cracks; $t_{0*} = 4.1$, $t_{1*} = 5.1$, $l_0^- = l_1^- = y_{0*} = y_{1*} = 0$, $l_1^+ = 20\Delta y$, $l_1^+ = 19\Delta y$. For $t_{0**} = 12.7$ and $t_{1**} = 13.8$ the cracks cease to grow, and the fracture ceases. The motion of the cracks occurs in jumps. We can follow this process using the example of the crack l_0. Fracture begins at $t = t_{0*}$ when the stresses $\tau_{0,0}$ exceed τ_p; a crack of length Δy is formed. Further, for $4.1 < t < 6.0$ the stresses $\tau_{0,y > \Delta y}$ remain less than τ_p, and the crack "stays" in its place. After this time the "breaking" peak of the stresses, reflected from a whole fiber and transferring part of its energy to it, arrives at the point $x = 0$, $y = \Delta y$, and the increase in the crack recommences. At subsequent instants of time ($t = 6.0 + n\Delta t$ ($n = 1, \ldots, 7$)) the "breaking" peaks arrive at points of the binder lying next to the broken fiber with coordinates $x = 0$, $y = (n + 1)\Delta y$, and the crack increases vigorously (with velocity c_1) to a value $l = 9\Delta y$ and for $t = 6.7$ ceases. Then when $t \geq 6.9$ it increases further to $l = 12\Delta y$. At this distance until the instant of time $t = 7.1$ the intensity of the "breaking" peak becomes less than τ_p, and the crack stays for $7.1 < t < 9.1$. For $t \geq 9.1$ a rapid increase in the crack again begins (with velocity c_1), and the process described is repeated until the crack finally does not stop: For $t > 12.7$ the shear stresses at the points $y > l_0^+$ do not reach the values τ_p. The crack l_1 moves in the same way as the increase in the crack l_0.

Two opposing factors affect the growth and cessation of the cracks: On the one hand, the stress peaks attenuate with distance from the fracture at the defect point, which helps the fracture to stop, and on the other hand, due to flaking, the average level of the stresses in the fibers increases, and the amplitudes of the peak values of the shear stresses in the depth of the binder increase. The fracture ceases when the first of these factors prevails.

Figure 3b shows the dependence of the fracture zone ($\bar{l} = (l_0^+ + l_1^-)/2$) of τ_p ($\alpha = 20$, $\Delta t = 0.1$). The curve shown is obtained by successive calculations for $1.1 \leq \tau_p \leq \tau_{max}$ ($\tau_{max} = 1.6$ are the maximum shear stresses in the binder) with a step τ_p of 0.05. The qualitative picture of the increase in the cracks for different τ_p agrees with that for $\tau_p = 1.25$. These calculations enable one to estimate the average velocity of motion of the cracks

$$V_{av} = \frac{1}{2}\left[\frac{l_0^+ - l_0^-}{t_{0**}} + \frac{l_1^+ - l_1^-}{t_{1**}}\right].$$

For the case $\alpha = 20$, for example, V_{av} decreases as τ_p increases. In the above range of values of τ_p the values of V_{av} vary respectively from $0.28\ c_1$ to $0.13\ c_1$.

It should be noted that the discrete scheme of the propagation of a fracture does not detect the occurrence or cessation of a fracture inside the range $0 < x < 1$ (case 3). This is due to the following features of the use of the fracture criterion in the numerical algorithm. Suppose that when $t = t^*$ the "breaking" peak reaches the point x^*, y ($0 < x^* < 1$). At the instant $t^* + \Delta t$ (in the discrete fracture scheme it occurs after a finite time of Δt) at this point the fracture conditions are realized, and since until the criterion is satisfied the grid functions are recalculated from time layer $t^* - \Delta t$ and t^* to the layer $t^* + \Delta t$, according to the wave equation (1), the peak considered at the instant of time $t = t^* + \Delta t$ appears at the point $x^* + \Delta x$, y (for motion of the wave in the positive direction) or at the point $x^* - \Delta x$, y otherwise. Hence, the fracture moves with unit velocity, not stopping until the fiber. Since the maximum peaks appear in the stress wave on reflection from the fibers, the fracture also begins at the join of the fiber and the binder.

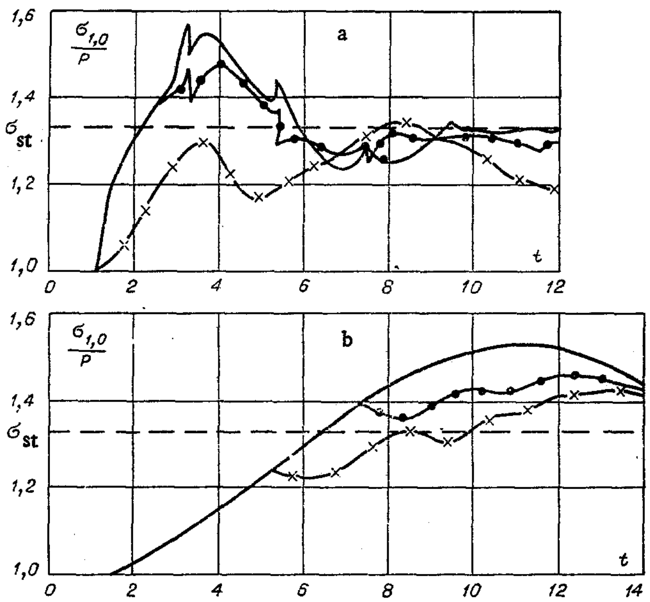

Fig. 4

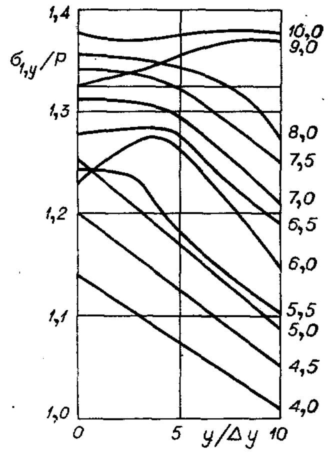

Fig. 5

A redistribution of the stretching–compression stresses in the fibers occurs during flaking (compared with the stresses when there is no fracture for $\tau_p > \tau_{max}$ $(t > 0)$): In the broken fiber $(j = 0)$ the stresses (compression) increase, and in the whole fibers they are reduced (stretching). If when $\tau_p > \tau_{max}$ in the first whole fiber the maximum loading is ~1.55 P and is reached at the point $y = 0$ [3], then for $\tau_p \le \tau_{max}$ when τ_p is reduced this value falls. In Fig. 4 we show values of the stretching stresses at the point $y = 0$ of the first whole fiber $(\sigma_{1,0})$, calculated for the following parameters: *a* – $\alpha = 2$, and $\Delta t = 0.05$, b – $\alpha = 20$, and $\Delta t = 0.1$. The continuous lines denote stresses when there is no flaking $(\tau_p > \tau_{max})$, the lines with the points are $\tau_p = 1.8$ and 1.4, and the lines with the crosses are $\tau_p = 1.35$ and 1.25 (Fig. 4a and b, respectively). The value $\sigma_{st} = (4/3)P$ corresponds to the value of $\sigma_{1,0}$ in the static problem [4] and is independent of α. When $\alpha = 20$ the maximum values of $\sigma_{1,0}$ are attained when $t = 12.5$ and 13.4 $(\tau_p = 1.4$ and 1.25, respectively).

It should be noted that during flaking the diagram of the stresses in the region $0 \le y \le l_1^+$ is straightened out, and fracture of the fiber (when the stresses reach a certain limiting value) is possible at any point of this range. Figure 5 shows, for $\alpha = 20$, $\Delta t = 0.1$ and $\tau_p = 1.35$, graphs of the stresses $\sigma_{1,y}$ at certain instants of time (the numbers on the right). It can be seen how the nature of the curves changes after the beginning of flaking $(t_{1*} = 5.1,\ l_1^- = 0,\ l_1^+ = 14\Delta y,\ t_{1**} = 12.2)$. Calculations showed that in the next whole fibers $(|j| > 1)$ the stressed state when there is flaking is practically the same as without it.

In [6] it was shown experimentally (the material of the fiber was glass and the filler was epoxy resin), that fracture of glass–plastic is initially accompanied by flaking cracks, after which at a certain distance (of the order of l_1^{+}) from the axis of symmetry (y = 0) breaking of a whole fiber occurs.

The results of the above calculations are in qualitative agreement with the conclusions obtained from experiments.

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FREE TORSIONAL OSCILLATIONS OF A STANDARD LINEAR BODY

P. M. Gorbunov

UDC 539.3

One of the problems of the torsional oscillations of a metal relaxing rod is considered in [1]. The behavior of the system in a time t is characterized by a function $\varphi(z, t)$, which defines the angle of rotation around the axis of the rod of an infinitely thin horizontal layer of material. The initial equation for the relaxation time $\tau \to \infty$ reduces to a wave-type equation which describes the motion of an idealized elastic material [2, 3].

However, the solution obtained in [1] as $\tau \to \infty$ is independent of the time, and hence does not agree with the solution of the similar problem for absolutely elastic materials [4]. This is due to the fact that when formulating the initial and boundary conditions in [1], zero initial values of the velocity and acceleration of the motion of the system were assumed for $t = 0$ over the whole specimen, whereas from the physical point of view motion of the system is only possible if its acceleration is different from zero.

We will consider the free torsional oscillations of a cylindrical uniform isotropic viscoelastic rod of radius R and length $h \gg 2R$, and a connected rigid disk. We will assume that the amplitude of the torsional oscillations of the distributed mass is small, the transverse cross sections $S(z)$ of the rod are not distorted, and are not displaced along the z axis ($S(z) = \text{const}$), and the torsion is not accompanied by a change in the volume of the deformed mass [1]. The z axis of a cylindrical system of coordinates (r, α, z) coincides with the axis of the rod. To determine the initial state of the system we will assume that before starting the pendulum the rod is twisted about the z axis by the continuous torsional moment of a pair of forces P concentrated on the boundary $S(z = h)$. Suppose that during a fairly large instant of time t_0 the rod reaches its initial statically loaded state. Then, for $t \geq t_0$ the torsional moment of the forces (PR_0) will be constant over the whole area of existence of the deformed mass $0 \leq z \leq h$, and is defined in the form

$$PR_0 = \frac{\mu\pi}{2} R^4 \partial\varphi(z)/\partial z, \tag{1}$$

where $\varphi(z)$ is the angle of rotation of the cross sections $S(z)$ (around the z axis) for a statically twisted state of the rod. If when $t_1 \geq t_0$ the forces P are simultaneously and instantaneously removed, the connected disk begins to change into a state of rotational motion around the z axis. We will assume that the relaxation

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